New Fusion Products For the Affine Kac-Moody Algebra $\widehat{\mathfrak{sl}}_2$ at Level $\kappa = 1/2$.

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Lie Algebras

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  - Anticommutative: $[x, y] = -[y, x]$
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Example

- $\mathfrak{gl}_n(\mathbb{C})$, $n \times n$ complex matrices
- $[a,b] = ab - ba$

Example

Abelian Lie Algebras: $\mathfrak{g} = V$, a vector space, with $[\cdot,\cdot] = 0$

Examples of Lie Algebras

Example

$\mathfrak{sl}_2(\mathbb{C})$: 2×2 complex matrices with trace 0.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Affine Kac-Moody Algebra $\widehat{\mathfrak{sl}}_2$

- Vector space over $\mathbb{C}$ with basis e_n, f_n, h_n and K for $n \in \mathbb{Z}$.

$$[h_m, e_n] = 2e_{m+n}, \quad [h_m, h_n] = 2m\delta_{m+n,0}K,$$
$$[e_m, f_n] = -h_{m+n} - m\delta_{m+n,0}K, \quad [h_m, f_n] = -2f_{m+n}.$$

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- A **representation** at level $\kappa = \frac{1}{2}$ is a vector space V and a Lie algebra homomorphism $\rho : \widehat{\mathfrak{sl}}_2 \to \mathrm{End}(V)$ such that K acts by $\frac{1}{2}\,\mathrm{id}$.
  - $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$.
  - $\rho(x)$ and $\rho(y)$ are linear maps $V :\to V$.

The Representations $\mathcal{L}_r$, for $r = 0, 1, 2, 3$

- $\mathrm{L}(r)$ is the finite dimensional irreducible $\mathfrak{sl}_2$-module of highest weight r. We have basis vectors $u_r, u_{r-2}, \ldots, u_{-r}$, such that

$$f_0 \cdot u_k = u_{k-2}.$$

- $\mathcal{U}\widehat{\mathfrak{sl}_2}^-$ has basis of monomials $\ldots f_{-n}^{a_{-n}} e_{-n}^{b_n} h_{-n}^{c_n} f_{-n+1}^{a_{-n-1}} e_{-n+1}^{b_{-n+1}} h_{-n+1}^{c_{-n+1}} \ldots$
- We form $\mathcal{U}\widehat{\mathfrak{sl}_2}^- \otimes \mathrm{L}(r)$ and this is an $\widehat{\mathfrak{sl}}_2$-module at level $\frac{1}{2}$. The zero grade subspace identifies with $\mathrm{L}(r)$.
- $\mathcal{U}\widehat{\mathfrak{sl}_2}^- \otimes \mathrm{L}(r)$ has a maximal proper subrepresentation and $\mathcal{L}_r$ is the quotient by it.

Singular Vectors

- A **singular vector** is a vector v in $\mathcal{U}\widehat{\mathfrak{sl}_2}^{-} \otimes \mathrm{L}(r)$ such that $e_n \cdot v = 0$ for all $n \geq 0$ and $f_n \cdot v = h_n \cdot v = 0$ for all $n \geq 1$.
- For $\mathcal{L}_3$, we calculate the singular vector to be

$$(-15e_{-2} + 6e_{-1}h_{-1})u_3 + (-4e_{-1}^2)u_1.$$

- Will be necessary to describe fusion products later on.

The Representations $\mathcal{E}_{\Lambda,\alpha}$

- $\mathsf{R}(\Lambda, \alpha)$ is generated by $v_{\Lambda+2\alpha}$.
- Like $\mathcal{L}_r$ except with an infinite zero grade, consisting of $v_{\Lambda+2\alpha+2n}$ for $n \in \mathbb{Z}$.
- $\mathcal{E}_{\Lambda,\alpha}$ is the quotient of $\mathcal{U}\widehat{\mathfrak{sl}_2}^{-} \otimes \mathsf{R}(\Lambda, \alpha)$ by its maximal proper submodule.
- We consider only $\Lambda = \frac{3}{2}$ and $\frac{5}{2}$ with $\alpha = \pm\frac{1}{4}$.

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- We consider only $\Lambda = \frac{3}{2}$ and $\frac{5}{2}$ with $\alpha = \pm\frac{1}{4}$.
- A **relaxed singular vector** is a vector $w \in \mathcal{E}_{\Lambda,\alpha}$ such that

$$f_n \cdot w = e_n \cdot w = h_n \cdot w = 0$$

  for all $n \geq 1$.
- For $\mathcal{E}_{\Lambda=5/2,\alpha}$, we calculate the relaxed singular vector to be

$$\left(e_{-1} f_0 - \alpha h_{-1} + \frac{2\alpha}{2\alpha + 5} f_{-1} e_0 \right) v_{\Lambda+2\alpha}.$$

Singular Vector Describing $\mathcal{E}_{3/2,\alpha}$

$$
\begin{aligned}
& e_{-3}f_0 + (-\frac{4}{3}\alpha - \frac{2}{3})h_{-3} + \frac{1+2\alpha}{2(1+\alpha)(2+\alpha)} f_{-2}f_{-1}e_0^2 + \frac{-\alpha - 1/2}{\alpha+1} f_{-2}h_{-1}e_0 \\
& + (\alpha + \frac{1}{2})f_{-2}e_{-1} - \frac{1}{\alpha - \frac{1}{2}} e_{-2}e_{-1}f_0^2 + e_{-2}h_{-1}f_0 \\
& - (\alpha + \frac{1}{2})e_{-2}f_{-1} - \frac{\frac{1}{2}\alpha + \frac{1}{4}}{\alpha+1} h_{-2}f_{-1}e_0 - \frac{1}{2}h_{-2}e_{-1}f_0 \\
& + (\frac{1}{2}\alpha + \frac{1}{4})h_{-2}h_{-1} + \frac{1+2\alpha}{12(1+\alpha)(2+\alpha)(3+\alpha)} f_{-1}^3 e_0^3 + \frac{1/2\alpha + 1/4}{\alpha+1} f_{-1}^2 e_{-1}e_0 \\
& - \frac{1+2\alpha}{4(1+\alpha)(2+\alpha)} f_{-1}^2 h_{-1} e_0^2 + \frac{2}{3(-3+2\alpha)(-1+2\alpha)} e_{-1}^3 f_0^3 + \frac{1}{2} f_{-1}e_{-1}^2 f_0 \\
& - \frac{\frac{1}{2}}{\alpha - 1/2} e_{-1}^2 h_{-1} f_0^2 - \frac{1}{6}(\alpha + \frac{1}{2})h_{-1}^3 + \frac{\frac{1}{2}\alpha + \frac{1}{4}}{\alpha+1} f_{-1}h_{-1}^2 e_0 + \frac{1}{2}e_{-1}h_{-1}^2 f_0 - (\alpha + \frac{1}{2})f_{-1}e_{-1}h_{-1}
\end{aligned}
$$

The Project: Computing Fusion Products

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- Fusion is an interesting algebraic operation for $\widehat{\mathfrak{sl}}_2$-modules $\mathcal{M}$ and $\mathcal{N}$ at a fixed level that resembles the tensor product and has important applications in two-dimensional Conformal Field Theory.
- The fusion product $\mathcal{M} \times \mathcal{N}$ is another $\widehat{\mathfrak{sl}}_2$-module at this level and $(\mathcal{K} \times \mathcal{M}) \times \mathcal{N} \cong \mathcal{K} \times (\mathcal{M} \times \mathcal{N})$ and $\mathcal{M} \times \mathcal{N} \cong \mathcal{N} \times \mathcal{M}$ holds. We have $\mathcal{L}_0 \times \mathcal{M} \cong \mathcal{M}$.
- It has been analyzed at positive integral level.
- Our project is to compute fusion products in the new case of modules at level $\frac{1}{2}$. (Ridout: $\kappa = -\frac{1}{2}$, Gaberdiel: $\kappa = -4/3$.)

Definition of Fusion Products

- Fusion product $\mathcal{M} \times \mathcal{N}$ is a quotient of $\mathcal{M} \otimes \mathcal{N}$.
- For $J = e, f$, or h, the action of J_n, denoted $\Delta(J_n)$, is given by the following rules:

$$\text{n} \geq 0 \quad \Delta(J_n) := \sum_{m=0}^{n} \binom{n}{m} J_m \otimes 1 + 1 \otimes J_n$$

$$\text{n} \geq 1 \quad \Delta(J_{-n}) := \sum_{m=0}^{\infty} \binom{n+m-1}{m} (-1)^m J_m \otimes 1 + 1 \otimes J_{-n}$$

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- The goal is to decompose these fusion products into direct sums. For example, representations $\mathcal{L}_r$ and $\mathcal{E}_{\Lambda,\alpha}$ that we already know may appear.

Fusion to Grade Zero

- Focus only on the zero grade of the representations $\mathcal{M}, \mathcal{N}$.
- Do this by setting all negative-grade expressions e_{-n}, f_{-n}, h_{-n} to zero.
- We use an algorithm to convert any $v \otimes w$ into zero-grade vectors:

$$\text{If } v = J_{-n}v' \text{ for some } n > 0, \text{ then } v \otimes w = -(-1)^n v' \otimes J_0 w.$$

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- By the above algorithm, singular vectors describing $\mathcal{M}$ and $\mathcal{N}$ give rise to vectors which must be set to 0 in the tensor product of the zero grades of $\mathcal{M}$ and $\mathcal{N}$.

Results of Fusion to Grade Zero

Theorem

1. $\mathcal{L}_3 \times \mathcal{L}_3 = \mathcal{L}_0$
2. $\mathcal{L}_3 \times \mathcal{L}_2 = \mathcal{L}_1$
3. $\mathcal{L}_3 \times \mathcal{L}_1 = \mathcal{L}_2$
4. $\mathcal{L}_2 \times \mathcal{L}_2 = \mathcal{L}_2 \oplus \mathcal{L}_0$
5. $\mathcal{L}_2 \times \mathcal{L}_1 = \mathcal{L}_3 \oplus \mathcal{L}_1$
6. $\mathcal{L}_1 \times \mathcal{L}_1 = \mathcal{L}_2 \oplus \mathcal{L}_0$.

1. $\mathcal{E}_{5/2,1/4} \times \mathcal{L}_3 = \mathcal{E}_{5/2,-1/4}$
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6. $\mathcal{E}_{3/2,1/4} \times \mathcal{L}_1 = \mathcal{E}_{3/2,-1/4} \oplus \mathcal{E}_{5/2,1/4}$.

Future Goals

- We are working on zero grade fusion of $\mathcal{E}_{\Lambda,\alpha}$ with $\mathcal{E}_{\Lambda',\alpha'}$.
- Compute fusion products to higher grades, since grade zero cannot fully describe fusion of $\mathcal{E}_{\Lambda,\alpha}$ with $\mathcal{E}_{\Lambda',\alpha'}$.
- Compute the full fusion products.
- See if the fusion operation closes on the representations we work with, which is important for conformal field theory.

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